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1987 J. Phys. A: Math. Gen. 20 3805

(http://iopscience.iop.org/0305-4470/20/12/029)

J. Phys. A: Math. Gen. 20 (1987) 3805-3823. Printed in the UK

Mode coupling between two waveguides with offset, tilt and gap using quantum theoretical methods

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Received 14 August 1986, in final form 30 December 1986

Abstract. Using quantum mechanical generalised coherent states, mode excitation by Gauss-Hermite or Gauss-Laguerre beams with spherical wavefronts is treated. For mode coupling between two waveguides with offset, tilt and gap recursion relations and sum rules are found.

1. Introduction

The mode structure of a waveguide field is an important characteristic of the waveguide. It is responsible for intermodal pulse dispersion, modal noise, radiation loss and so on (Grau 1981, Snyder and Love 1983). If the distance from the radiating source does not exceed a characteristic length sufficient for the establishment of an equilibrium modal distribution in the waveguide, then the mode structure of the field depends only on the excitation conditions. The waveguides in fibre optics and in integrated optics are often excited by radiation from laser sources or by beams from the end face of another irradiating waveguide. The modes of typical laser beams and the modes of parabolic-index waveguides are usually described by Gauss-Hermite or Gauss-Laguerre functions. It is known (Yariv 1975) that such beams do not change their functional form during propagation in free space: only their width, angular divergence and wavefront curvature are changing. Moreover, it is known (Marcuse 1972, Krivoshlykov *et al* 1985a) that fundamental modes of active parabolic-index waveguides (with gain) are also described by Gaussian beams with spherical wavefronts, whose radii of curvature are related to the waveguide gain parameter.

The aim of this paper is to investigate the mode excitation coefficients for a multimode parabolic-index optical waveguide excited by a Gauss-Hermite or Gauss-Laguerre beam with spherical wavefront. For the sake of definiteness, we shall concentrate in this paper on the problem of mode coupling between two multimode parabolic-index waveguides, keeping in mind that all results obtained may also be used for the description of waveguide mode excitation by laser beams. In an ideal case, both waveguides would be identical, without axial distance, without tilt of axes and without transversal offset. The mode distributions in both waveguides would coincide. In reality, the two waveguides are different and the connection is imperfect. This will introduce a changed mode distribution in the second waveguide by coupling between the modes of the guides.

0305-4470/87/123805+19$02.50

The effect of these imperfections has been treated in some way or other by many authors. We mention only the following papers. The coupling coefficients between fundamental modes of planar graded-index waveguides were obtained by Marcuse (1977) and turn out to be similar to those between Gaussian beams and the fundamental mode of a lens waveguide obtained by Kogelnik (1964). Mode excitation of a multimode parabolic-index waveguide by coherent Gaussian beams was investigated by Grau *et al* (1980), Saijonmaa *et al* (1980) and Georg (1982). The similar problem for the case of excitation by partially coherent Gaussian beams was solved by Krivoshlykov *et al* (1985b). Coupling coefficients between modes of lenslike media were obtained by Arnaud (1971). In Krivoshlykov and Sissakian (1979) algebraic and group theoretic methods were developed to investigate the mode coupling between two butt-jointed parabolic-index optical waveguides, and in Krivoshlykov *et al* (1983b) these methods were used to investigate mode coupling in the case when two different general square-law-index multimode waveguides with elliptical cross sections are to be butt-jointed with offset and tilt. The advantage of the algebraic approach is that it provides a uniform treatment of all the mentioned effects and that it allows explicit expressions to be obtained for the coupling coefficients in the most general case. In the present paper we shall generalise this approach for calculating mode coupling coefficients in the most general case when there is even a longitudinal gap between the waveguides which are to be connected.

We assume both guides showing different parabolic refractive index distributions extending to infinity. In this way we approximate real multimode waveguides with gradient index carrying a large number of modes. The mode fields are solutions of the scalar Helmholtz equation, which—in this case—is identical with the time-independent two-dimensional Schrödinger equation. In abstract language: one has to solve the quantum mechanical eigenvalue problem for the two-dimensional harmonic oscillator. We use the well known operator methods of quantum theory and introduce annihilation and creation operators for both waveguides.

By a succession of four linear canonical transformations we can describe the offset, the tilt (through an infinitesimally small angle) and the axial gap between the different waveguides and obtain a relationship between the operators of both waveguides. We obtain a set of recurrence relations between the coupling coefficients. To start these relations only a few simple Gaussian integrals have to be solved. For the canonical transformation describing the axial gap, we use generalised coherent states—a generalisation of the usual coherent states which are known in quantum mechanics as correlated coherent states (Dodonov *et al* 1980). It was shown by Krivoshlykov *et al* (1983a) that these generalised coherent states describe Gaussian beams with spherical wavefronts. In Krivoshlykov and Sissakian (1986) the generalised Fock representation of occupation numbers and the generalised angular momentum representation were introduced and it was shown that these states represent spherical Gauss–Hermite or Gauss–Laguerre beams which are useful to describe beams with curved wavefronts at the front plane of the second waveguide. We represent the solution in cartesian coordinates using Gauss–Hermite functions or in cylindrical coordinates using Gauss–Laguerre functions.

In § 2, we connect the optical problem with quantum theory. Section 3 gives the coordinate transformations, resulting from the geometrical imperfections: axial shift, rotation and transversal shift. We also define the mode coupling coefficients and the mode overlap integrals in the general (transversally) two-dimensional case for two different but circular-symmetric fibres. Section 4 specialises to the case of only one transversal dimension (slab waveguide) and describes the four canonical transformations. In § 5, we derive the recurrence relations for the overlap integrals and the pertinent integrals necessary for starting them. In § 6, the coupling between Gauss-Laguerre modes of coaxial circular-symmetric fibres is investigated. The appendices 1-3 give a short review of this new type of generalised states because the original papers are not available in English.

2. Connection between the optical problem and quantum theory

We consider a dielectric circular-symmetric waveguide showing a parabolic distribution of the refractive index (homogeneous in the z direction):

$$n^2(x_1, x_2) = n^2(r) = \begin{cases} n_0^2 - \omega^2 r^2 = n_0^2[1 - 2\Delta(r/a)^2] & \text{for } r < a \\ n_0^2 - \omega^2 a^2 & \text{for } r > a \end{cases} \tag{1}$$

where $r = (x_1^2 + x_2^2)^{1/2}$. As usual (Grau 1981, Snyder and Love 1983), we have defined the relative difference of refractive indices

$$\Delta = (\omega^2 a^2)/(2n_0^2)$$

and we set

$$V = n_0 k a (2\Delta)^{1/2} = k a^2 \omega$$

where $k = 2\pi f/c$. The parameter $\omega (\neq 2\pi f)$ is the gradient constant of the waveguide with profile (1) and f is the light frequency.

The parabolic profile will now be extended to infinity in the transversal x_1, x_2 plane (no truncation at $r = a$). The time dependence is chosen as $\exp(-2\pi i f t)$. For the solution of the scalar Helmholtz equation we use the ansatz

$$E(x_1, x_2, z) = \psi(x_1, x_2) \exp(i\beta z) \tag{2}$$

where $\psi(x_1, x_2)$ is the solution of the following eigenvalue problem (Krivoshlykov and Sissakian 1979):

$$\left(-\frac{1}{2k^2}(\partial_1^2 + \partial_2^2) + \frac{n_0^2 - n^2}{2} \right) \psi(x_1, x_2) = \varepsilon \psi(x_1, x_2) \tag{3}$$

where we set $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$ and

$$\beta/k = (n_0^2 - 2\varepsilon)^{1/2}. \tag{4}$$

We introduce the momentum operators $\hat{p}_j = -(i/k)\partial_j$ and the operators of position $\hat{x}_j$ (multiplication by x_j) in coordinate space representation (Marcuse 1972). (We do not distinguish between operators in abstract Hilbert space and operator representations in some function space.) Then

$$[\hat{x}_i, \hat{p}_j] = (i/k)\delta_{ij}\hat{I}$$

($\hat{I}$ is the unit operator) and

$$\hat{H}\psi = \varepsilon\psi \tag{5}$$

with the Hamiltonian

$$\hat{H} = \tfrac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2) + \tfrac{1}{2}\omega^2(\hat{x}_1^2 + \hat{x}_2^2) = \frac{\omega}{k}\sum_1^2 \left(\hat{a}_j^+ \hat{a}_j + \frac{\hat{I}}{2} \right). \tag{6}$$

Here, the $\hat{a}_j$, $\hat{a}_j^+$ are annihilation and creation operators which satisfy the commutation relations $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}\hat{I}$. They are defined as follows:

$$\begin{pmatrix} \hat{a}_j \\ \hat{a}_j^+ \end{pmatrix} = \left(\frac{k}{2}\right)^{1/2} \begin{pmatrix} \omega^{1/2} & \mathrm{i}\omega^{-1/2} \\ \omega^{1/2} & -\mathrm{i}\omega^{-1/2} \end{pmatrix} \begin{pmatrix} \hat{x}_j \\ \hat{p}_j \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (k\omega)^{1/2} & (k\omega)^{-1/2} \\ (k\omega)^{1/2} & -(k\omega)^{-1/2} \end{pmatrix} \begin{pmatrix} \hat{x}_j \\ \partial_j \end{pmatrix} \tag{7}$$

or

$$\begin{pmatrix} \hat{x}_j \\ \partial_j \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (k\omega)^{-1/2} & (k\omega)^{-1/2} \\ (k\omega)^{1/2} & -(k\omega)^{1/2} \end{pmatrix} \begin{pmatrix} \hat{a}_j \\ \hat{a}_j^+ \end{pmatrix}. \tag{8}$$

If we replace $1/k$ by $\hbar$ (Planck's constant/2π), then (5) and (6) describe the quantum mechanical eigenvalue problem for a two-dimensional harmonic oscillator and (3) is the stationary Schrödinger equation for this potential.

The solution of (5) is known (Landau and Lifshitz 1965); the eigenvalues are

$$\varepsilon_{m_1 m_2} = (\omega/k)(m_1 + m_2 + 1) \qquad m_1, m_2 \geqslant 0, \text{ integer.} \tag{9}$$

With (4) we obtain for the propagation constant

$$\beta_{m_1 m_2} = kn_0(1 - 2\varepsilon_{m_1 m_2}/n_0^2)^{1/2} \tag{10a}$$

$$\approx kn_0 - k\varepsilon_{m_1 m_2}/n_0 = \left(\frac{kn_0}{2} - \frac{\omega}{n_0}(m_1 + \tfrac{1}{2})\right) + \left(\frac{kn_0}{2} - \frac{\omega}{n_0}(m_2 + \tfrac{1}{2})\right) \equiv \beta_{m_1} + \beta_{m_2} \tag{10b}$$

$$\approx kn_0/2 + kn_0/2 = n_0 k \tag{10c}$$

(the first approximation holds for $n_0^2 \gg \varepsilon_{m_1 m_1}$; the second one holds for $kn_0^2 \gg \omega m_j$ and gives a β value independent from m_1, m_2).

The guided modes of the waveguide are the eigenfunctions (Gauss-Hermitian functions) of the eigenvalue equation (5). We write them as

$$\psi_{m_1 m_2}(x_1, x_2) = \langle x_1 x_2 | m_1 m_2 \rangle$$

which coincides with (A2.4) for $\omega_1 = \omega_2 = \omega$. The coherent states in coordinate space representation:

$$\psi_{\alpha_1 \alpha_2}(x_1, x_2) = \langle x_1 x_2 | \alpha_1 \alpha_2 \rangle$$

coincide with (A1.5) for $\omega_1 = \omega_2 = \omega$ and represent the best approximation to a geometrical ray in the waveguide (1) (Krivoshlykov and Sissakian 1979).

Finally, we now write (cf (2))

$$E_{m_1 m_2}(x_1, x_2, z) = \langle x_1 x_2 | \overline{m_1 m_2} \rangle \tag{11}$$

with

$$|\overline{m_1 m_2}\rangle = \exp(\mathrm{i}\beta_{m_1 m_2} z)|m_1 m_2\rangle = O_{m_1 m_2}(z)|m_1 m_2\rangle. \tag{12}$$

Analogously, we define, using (10c), the coherent states $|\overline{\alpha_1 \alpha_2}\rangle = \exp(\mathrm{i}kn_0 z)|\alpha_1 \alpha_2\rangle$.

All these relations hold for the first waveguide as well as for the second but all quantities referring to the second waveguide will be primed in the following (e.g. ω and ω', moreover eigenvalues m_1, m_2 (respectively n_1', n_2'), but we set $n_0 = n_0'$ in (1)). For the connection between the annihilation and creation operators $\hat{a}_j$, $\hat{a}_j^+$ and $\hat{a}_j'$, $\hat{a}_j'^+$ of both waveguides we shall find a canonical transformation.

3. The transformation of coordinates and the mode overlap integrals

The coordinate systems of both waveguides are connected by a translation in longitudinal direction (gap), by a shift in transversal direction (offset) and by an infinitesimal rotation of the waveguide axes.

We shall distinguish two different systems of rectangular coordinates (figure 1), the z, z' axes of which show along the waveguide axes and the x_2, x_2' axes are parallel to the line of intersection of the end plane of waveguide 1 with the front plane of waveguide 2. The coordinate system Ox_1x_2z and the auxiliary system Px_1x_2z are related by a shift $z \to z + z_0$. The coordinate systems Px_1x_2z and $O'x_1'x_2'z'$ are related by a shift of the centres from P to O' (the distance d has been decomposed into $d_{x_1'} \equiv d_1$ and $d_{x_2'} \equiv d_2$ along the x_1', x_2' axes), followed by a rotation through the angle θ about the x_2' axis:

$$\begin{aligned} x_1 &= (x_1' - d_1)\cos\theta - z'\sin\theta \\ x_2 &= x_2' - d_2 \\ z &= (x_1' - d_1)\sin\theta + z'\cos\theta. \end{aligned} \tag{13}$$

The end face of waveguide 1 irradiates Gauss-Hermite modes with plane wavefronts $|m_1m_2\rangle$ (A2.2), which are generated by the operator $\hat{a}_j$ (A1.3) and are described by $\psi_{m_1m_2}(x_1, x_2)$ (A2.4) with $\omega_1 = \omega_2 = \omega$. The initial spot size at $z = 0$ is $\sigma_0 = 1/(2\omega k)^{1/2}$. After propagating the distance z_0 in free space, we get at point P a radius $R(z)$ of curvature and a width $\sigma(z)$ of a fundamental Gaussian beam $|\underline{00}\rangle$ (Yariv 1975)

$$R(z = z_0) = z_0\left(1 + \frac{1}{\omega^2 z_0^2}\right) \qquad \sigma^2(z_0) = \sigma_0^2(1 + z_0^2\omega^2). \tag{14}$$

From $R(z_0)$, $\sigma(z_0)$, we find—according to (A1.16)—the parameters μ and χ of the generalised operator $\underline{\hat{a}}_j$ (A1.12), which generates spherical Gauss-Hermite modes $|\underline{m}_1\underline{m}_2\rangle$ (A2.7). These modes may be represented by a generalised Fock's representation of occupation numbers $\underline{\psi}_{m_1m_2}(x_1, x_2)$ (A2.8). In the next section, we derive the canonical transformation between these operators $\underline{\hat{a}}_j$, $\underline{\hat{a}}_j^+$ and the usual operators $\hat{a}_j'$, $\hat{a}_j'^+$ of the second waveguide, using the transformation (13) of the coordinates.

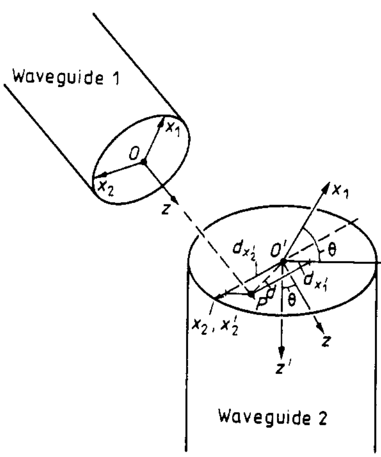

Figure 1. Geometry for the general transformations with gap $OP = z_0$, tilt through θ, offset PO'.

The mode coupling coefficient between a mode m_1, m_2 of the first waveguide and a mode n_1', n_2' of the second waveguide is given by $|T^{n_1'n_2'}_{m_1m_2}|^2$, where

$$T^{n_1'n_2'}_{m_1m_2} = \langle \overline{n_1'n_2'}|\overline{\underline{m}_1\underline{m}_2}\rangle = \langle n_1'n_2'|O^+_{n_1'n_2'}(z')O_{m_1m_2}(z)|\underline{m_1m_2}\rangle|_{z'=0}$$

$$= \iint \psi^*_{n_1'n_2'}(x_1', x_2') \exp[-\mathrm{i}\beta_{n_1'n_2'}z' + \mathrm{i}\beta_{m_1m_2}z(x_1', z')]$$

$$\times \underline{\psi}_{m_1m_2}[x_1(x_1', z'), x_2(x_2')]\,\mathrm{d}x_1'\,\mathrm{d}x_2'|_{z'=0}$$

$$= \iint \psi^*_{n_1'n_2'}(x_1', x_2') \exp[\mathrm{i}kn_0(x_1' - d_1)\sin\theta]$$

$$\times \underline{\psi}_{m_1m_2}[x_1(x_1'), x_2(x_2')]\,\mathrm{d}x_1'\,\mathrm{d}x_2'$$

$$= \langle \tilde{n}_1'|\underline{\tilde{m}}_1\rangle\langle \tilde{n}_2'|\underline{\tilde{m}}_2\rangle = T^{n_1'}_{m_1}T^{n_2'}_{m_2} \qquad (15)$$

is the mode overlap integral in the front plane of the second waveguide: $z' = 0$. Here, equations (10c), (12) and (13) have been used.

These mode overlap integrals are the mode coupling coefficients between the modes of the different waveguides. They are at the same time the excitation coefficients for the second waveguide. The measurements (Bartelt *et al* 1983, Golub *et al* 1984, see also Shigesawa *et al* 1978) are in agreement with theory (when $z_0 = 0$). The experimental set-up is also described in these papers.

We see from (15) that, in the case of waveguides with circular symmetry, the mode overlap integrals may be reduced to the product of two one-dimensional integrals, if an appropriate coordinate system is chosen. Therefore, it is sufficient to study only the one-dimensional problem.

4. Canonical transformations in the case of parabolic-index slab waveguides

Consider mode coupling between two one-dimensional slab waveguides (figure 2). The modes with plane wavefronts coming from the first waveguide are generated by the operators $\hat{a}_j$ (7) with $j = 1$ (this subscript 1 will be omitted in the following). The case $j = 2$ is mentioned at the end of § 5. After propagating a distance z_0 in free space, they transform into spherical beams, which are generated by the generalised operators $\underline{\hat{a}}$ (A1.12), where the parameters μ and χ are connected with the graded-index waveguide parameter ω and the distance z_0 between both waveguides according to

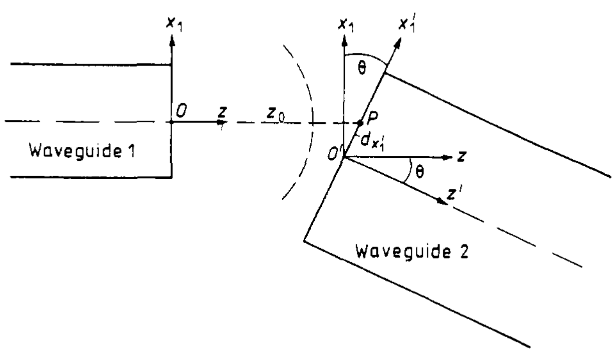

Figure 2. Geometry for the transformation for two slab waveguides.

(A1.16) and (14). This translation from $z=0$ to z_0 is expressed by the homogeneous canonical transformation (A1.23) between $\hat{a}$ and $\hat{\underline{a}}$. The modes of the second waveguide are generated by the operator $\hat{a}'$ ((7), where we should replace ω by ω').

Now introducing the transformation (13) into $\hat{\underline{a}}$, we obtain the relations

$$\begin{aligned} \hat{\underline{a}} &= \tau\hat{a}' + \eta\hat{a}'^{+} - \delta & \qquad \hat{a}' &= \tau^*\hat{\underline{a}} - \eta\hat{\underline{a}}^{+} + \zeta \\ \hat{\underline{a}}^{+} &= \eta^*\hat{a}' + \tau^*\hat{a}'^{+} - \delta^* & \qquad \hat{a}'^{+} &= -\eta^*\hat{\underline{a}} + \tau\hat{\underline{a}}^{+} + \zeta^* \end{aligned} \tag{16}$$

where

$$\begin{aligned} \left.\begin{matrix}\tau\\ \eta\end{matrix}\right\} &= \frac{1}{2(\cos\chi)^{1/2}} \cdot \left(\frac{1}{(\mu\omega')^{1/2}} \pm (\mu\omega')^{1/2}\exp(\mathrm{i}\chi)\right) \\ \delta &= \left(\frac{k}{2\mu\cos\chi}\right)^{1/2} [d + \mathrm{i}\mu n_0\theta\exp(\mathrm{i}\chi)] \\ \zeta &= (k\omega'/2)^{1/2}(d + \mathrm{i}n_0\theta/\omega') = \zeta_d + \zeta_\theta. \end{aligned} \tag{17}$$

Here, we took into account the z' dependence of the field in the second waveguide: $\exp(\mathrm{i}\beta' z')$, which gives

$$\partial_{z'} = \mathrm{i}\beta' \approx \mathrm{i}\beta \approx \mathrm{i}kn_0 \tag{18}$$

and supposed that only very small angles θ between waveguide axes are considered: $\cos\theta \approx 1$, $\sin\theta \approx \theta$. Furthermore, the plane $z'=0$ of the second waveguide is considered.

The transformation from $\hat{a}$, $\hat{a}^{+}$ to $\hat{a}'$, $\hat{a}'^{+}$ altogether is a linear inhomogeneous canonical transformation T:

$$\begin{pmatrix}\hat{a}'\\ \hat{a}'^{+}\\ 1\end{pmatrix} = T\begin{pmatrix}\hat{a}\\ \hat{a}^{+}\\ 1\end{pmatrix}$$

where $T = T_4T_3T_2T_1$ is the product of four successive canonical transformations:

$$T_1 = \begin{pmatrix} u & v & 0\\ v^* & u^* & 0\\ 0 & 0 & 1\end{pmatrix} \qquad T_2 = \begin{pmatrix} \tau^* & -\eta & 0\\ -\eta^* & \tau & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1 & 0 & \zeta_d\\ 0 & 1 & \zeta_d^*\\ 0 & 0 & 1\end{pmatrix} \qquad T_4 = \begin{pmatrix} 1 & 0 & \zeta_\theta\\ 0 & 1 & \zeta_\theta^*\\ 0 & 0 & 1\end{pmatrix}.$$

Here, T_1 (from (A1.23)) gives the translation from the end face of waveguide 1 at $z=0$ to the front face of waveguide 2 at $z=z_0$. With T_2 (where $|\tau|^2 - |\eta|^2 = 1$), the beam is matched to the waveguide 2 (transition from the parameters μ, χ to the parameter ω'). The following transformation T_3 is causing the transversal shift = offset through the distance d between P and O' (figure 1), and T_4 represents the rotation through the infinitesimal angle θ about the x_2' axis. We see that this rotation T_4 corresponds to a transversal shift (T_3), but the real distance d must be replaced by the imaginary quantity $\mathrm{i}n_0\theta/\omega'$. The transformations T_3, T_4 represent two purely inhomogeneous commuting transformations. In (16), we have taken into account only the transformation $T_4T_3T_2$, while T_1 has already been considered by using the generalised Gauss–Hermitian beams with spherical wavefronts.

5. Recurrence relations and start integrals for slab waveguides

The mode overlap integrals are the matrix elements (cf (15))

$$T_m^{n'} = \langle \tilde{n}'|\tilde{m}\rangle = \int \psi_{n'}^*(x')\exp[ikn_0(x'-d)\theta]\psi_m[x(x')]\,dx' \tag{19}$$

where

$$\hat{a}'^+\hat{a}'|n'\rangle = n'|n'\rangle \qquad \underline{\hat{a}}^+\underline{\hat{a}}|\underline{m}\rangle = \underline{m}|\underline{m}\rangle$$

$$\hat{a}'|\alpha'\rangle = \alpha'|\alpha'\rangle \qquad \underline{\hat{a}}|\underline{\alpha}\rangle = \underline{\alpha}|\underline{\alpha}\rangle.$$

In coordinate space representation

$$\langle x'|n'\rangle = \psi_{n'}(x') = \left(\frac{\omega'k}{\pi}\right)^{1/4}(2^{n'}n'!)^{-1/2}\exp(-\omega'kx'^2/2)H_{n'}(x'(\omega'k)^{1/2})$$

$$\langle x'|\underline{m}\rangle = \underline{\psi}_m(x') = \left(\frac{k\cos\chi}{\pi\mu}\right)^{1/4}\frac{\exp(-im\chi)}{(2^m m!)^{1/2}}\exp\left(-\exp(-i\chi)\frac{kx'^2}{2\mu}\right)H_m\left(x'\left(\frac{(k\cos\chi)}{\mu}\right)^{1/2}\right) \tag{20}$$

$$\langle x'|\gamma'\rangle = \psi_{\gamma'}(x') = \left(\frac{\omega'k}{\pi}\right)^{1/4}\exp\left(-\omega'kx'^2/2+(2\omega'k)^{1/2}\gamma'x'-\gamma'^2/2-|\gamma'|^2/2\right)$$

$$\langle x'|\underline{\alpha}\rangle = \underline{\psi}_\alpha(x') = \left(\frac{k\cos\chi}{\pi\mu}\right)^{1/4}\exp\left\{-\exp(-i\chi)\left[x'\left(\frac{k}{2\mu}\right)^{1/2}-\underline{\alpha}(\cos\chi)^{1/2}\right]^2+\underline{\alpha}^2/2\right.$$
$$\left.+\underline{\alpha}^2/2-|\underline{\alpha}|^2/2\right\}.$$

Calculating $\langle n'|\hat{a}'|\underline{m}\rangle$, $\langle n'|\underline{\hat{a}}|\underline{m}\rangle$, $\langle n'|\hat{a}'^+|\underline{m}\rangle$ and $\langle n'|\underline{\hat{a}}^+|\underline{m}\rangle$ yields the following recurrence relations

$$T_{m+1}^{n'} = \frac{\tau^*}{\eta}\left(\frac{m}{m+1}\right)^{1/2}T_{m-1}^{n'} - \frac{1}{\eta}\left(\frac{n'+1}{m+1}\right)^{1/2}T_m^{n'+1} + \frac{\zeta}{\eta}\left(\frac{1}{m+1}\right)^{1/2}T_m^{n'}$$

$$T_m^{n'+1} = -\frac{\eta}{\tau}\left(\frac{n'}{n'+1}\right)^{1/2}T_m^{n'-1} + \frac{1}{\tau}\left(\frac{m}{n'+1}\right)^{1/2}T_{m-1}^{n'} + \frac{\delta}{\tau}\left(\frac{1}{n'+1}\right)^{1/2}T_m^{n'}$$

$$T_{m+1}^{n'} = \frac{\eta^*}{\tau}\left(\frac{m}{m+1}\right)^{1/2}T_{m-1}^{n'} + \frac{1}{\tau}\left(\frac{n'}{m+1}\right)^{1/2}T_m^{n'-1} - \frac{\zeta^*}{\tau}\left(\frac{1}{m+1}\right)^{1/2}T_m^{n'} \tag{21}$$

$$T_m^{n'+1} = -\frac{\tau^*}{\eta^*}\left(\frac{n'}{n'+1}\right)^{1/2}T_m^{n'-1} + \frac{1}{\eta^*}\left(\frac{m+1}{n'+1}\right)^{1/2}T_{m+1}^{n'} + \frac{\delta^*}{\eta^*}\left(\frac{1}{n'+1}\right)^{1/2}T_m^{n'}.$$

A graphical representation is shown in figure 3. If three of the points (matrix elements) of such a graph are known, the fourth can easily be found, and only a restricted number of matrix elements must be known explicitly (for example T_0^0, T_1^0, T_2^0, T_3^0 or T_0^0, T_0^1, T_0^2, T_0^3). In the following, we calculate T_0^0, T_m^0, $T_0^{n'}$ (terms with θ^2 in the exponent will be neglected compared to θ). Using (20), we obtain immediately

$$T_0^0 = \langle \widetilde{n'=0}|\widetilde{\underline{m}=0}\rangle = \langle \widetilde{\gamma'=0}|\widetilde{\underline{\alpha}=0}\rangle$$

$$= \int \psi_{\gamma'=0}^*(x')\exp[ikn_0(x'-d)\theta]\underline{\psi}_{\alpha=0}[x(x')]\,dx'$$

$$= \left(\frac{2(\mu\omega' \cos\chi)^{1/2}}{\omega'\mu + \exp(-i\chi)} \right)^{1/2} \exp\left(-\frac{\omega' k \exp(-i\chi)}{2[\omega'\mu + \exp(-i\chi)]} d^2 - \frac{ikn_0 d\omega'\mu}{\omega'\mu + \exp(-i\chi)} \theta \right) \tag{22}$$

and

$$\langle \tilde{\gamma}' | \tilde{\underline{\alpha}} \rangle = \int \psi^*_{\gamma'}(x') \exp[ikn_0(x'-d)\theta] \underline{\psi}_\alpha[x(x')]\, dx'$$

$$= T_0^0 \exp\left(-\frac{|\underline{\alpha}|^2 + |\gamma'|^2}{2} \right) \exp\left(-\frac{\exp(-i\chi)(2k\mu\cos\chi)^{1/2}}{\omega'\mu + \exp(-i\chi)} (\omega' d - in_0\theta)\underline{\alpha} \right.$$
$$+ \frac{(2\omega' k)^{1/2}}{\omega'\mu + \exp(-i\chi)} (d\exp(-i\chi) + i\mu n_0\theta)\gamma'^*$$
$$+ \frac{2(\omega'\mu\cos\chi)^{1/2}}{\omega'\mu + \exp(-i\chi)} \exp(-i\chi)\underline{\alpha}\gamma'^*$$
$$\left. + \frac{1}{2}\frac{\omega'\mu - \exp(-i\chi)}{\omega'\mu + \exp(-i\chi)} \gamma'^{*2} + \frac{1}{2}\frac{\exp(-i\chi)[1-\omega'\mu\exp(-i\chi)]}{\omega'\mu + \exp(-i\chi)} \underline{\alpha}^2 \right). \tag{23}$$

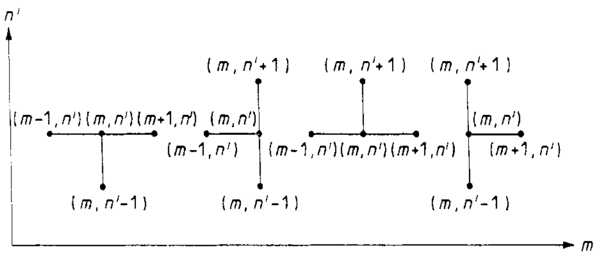

Figure 3. Graphical representation of the recurrence relations (21).

If we use (23) as a generating function for mode overlap integrals, then, according to (A2.3), we obtain

$$T_m^{n'} = T_0^0 (m!\,n'!)^{-1/2} H_{mn'}(C, \xi_1, \xi_2) \tag{24}$$

with

$$\xi_1 = -\frac{(2k\mu\cos\chi)^{1/2}}{1+\omega'\mu\exp(-i\chi)} (\omega' d + in_0\theta)$$

$$\xi_2 = \frac{(2\omega' k)^{1/2}}{1+\omega'\mu\exp(-i\chi)} [d - i\mu n_0\theta\exp(-i\chi)]$$

$$C = \begin{bmatrix} -\dfrac{\exp(-i\chi)[1-\omega'\mu\exp(-i\chi)]}{\omega'\mu+\exp(-i\chi)} & -2\dfrac{(\omega'\mu\cos\chi)^{1/2}}{\omega'\mu+\exp(-i\chi)}\exp(-i\chi) \\ -2\dfrac{(\omega'\mu\cos\chi)^{1/2}}{\omega'\mu+\exp(-i\chi)}\exp(-i\chi) & -\dfrac{\omega'\mu-\exp(-i\chi)}{\omega'\mu+\exp(-i\chi)} \end{bmatrix}$$

and $H_{mn}(C, \xi_1, \xi_2)$ is a two-dimensional Hermite polynomial (Erdelyi 1953).

The general formula (24) is not very convenient due to the Hermite polynomials H_{mn}. The special mode overlap integrals between the Gaussian fundamental mode of the second (first) waveguide and modes of the first (second) waveguide, however, may be expressed in terms of the usual one-dimensional Hermite polynomials.

From (23), we obtain

$$\langle \widetilde{\gamma'=0}|\tilde{\underline{\alpha}}\rangle = T_0^0 \exp(-|\underline{\alpha}|^2/2) \exp\left(-\frac{\exp(-i\chi)(2k\mu \cos\chi)^{1/2}}{\omega'\mu+\exp(-i\chi)}(\omega' d - i n_0\theta)\underline{\alpha} + \frac{1}{2}\frac{\exp(-i\chi)[1-\omega'\mu\exp(-i\chi)]}{\omega'\mu+\exp(-i\chi)}\underline{\alpha}^2\right)$$

or, with

$$|\alpha\rangle = \exp(-|\underline{\alpha}|^2/2)\sum_m (m!)^{-1/2}\underline{\alpha}^m|\underline{m}\rangle$$

$$\langle \widetilde{\gamma'=0}|\tilde{\underline{\alpha}}\rangle = \exp(-|\underline{\alpha}|^2/2)\sum_m (m!)^{-1/2}\underline{\alpha}^m\langle\tilde{0}|\tilde{m}\rangle$$

$$= \exp(-|\underline{\alpha}|^2/2)\sum_m (m!)^{-1/2}\underline{\alpha}^m T_m^0.$$

Comparing both expressions and using the generating function for Hermitian polynomials, we have

$$T_m^0 = T_0^0\frac{\exp(-im\chi)}{(2^m m!)^{1/2}}\left(\frac{\omega'\mu-\exp(-i\chi)}{\omega'\mu+\exp(-i\chi)}\right)^{m/2} \times H_m\left(\left(\frac{k\mu\cos\chi}{\omega'^2\mu^2-\exp(-2i\chi)}\right)^{1/2}(in_0\theta-\omega' d)\right). \tag{25}$$

Correspondingly

$$\langle\tilde{\gamma}'|\underline{\widetilde{\alpha=0}\leftrightarrow\widetilde{m=0}}\rangle$$

$$= T_0^0\exp(-|\gamma'|^2/2)\exp\left(\frac{(2\omega' k)^{1/2}}{\omega'\mu+\exp(-i\chi)}[d\exp(-i\chi)+i\mu n_0\theta]\gamma'^* + \frac{1}{2}\frac{\omega'\mu-\exp(-i\chi)}{\omega'\mu+\exp(-i\chi)}\gamma'^{*2}\right)$$

$$= \exp(-|\gamma'|^2/2)\sum_{n'}(n'!)^{-1/2}(\gamma'^*)^{n'}\langle\tilde{n}'|\tilde{\underline{0}}\rangle$$

$$= \exp(-|\gamma'|^2/2)\sum_{n'}(n'!)^{-1/2}(\gamma'^*)^{n'}T_0^{n'}.$$

Therefore

$$T_0^{n'} = T_0^0(2^{n'}n'!)^{-1/2}\left(\frac{\exp(-i\chi)-\omega'\mu}{\exp(-i\chi)+\omega'\mu}\right)^{n'/2}H_{n'}(\tilde{u}) \tag{26}$$

where

$$\tilde{u} = \left(\frac{\omega' k}{\exp(-2i\chi)-\omega'^2\mu^2}\right)^{1/2}[d\exp(-i\chi)+i\mu n_0\theta].$$

The quantities $q(n', m) = |T_m^{n'}/T_0^0|^2$ are relative intensities and can be measured (Golub 1984). We find the following relations between $q(1,0)$, $q(2,0)$ and the parameters d, θ, μ, ω'

$$q(1,0) = 2\left|\frac{\exp(-\mathrm{i}\chi) - \omega'\mu}{\exp(-\mathrm{i}\chi) + \omega'\mu}\right| |\tilde{u}|^2$$
$$= 2\omega' k \left|\frac{d \exp(-\mathrm{i}\chi) + \mathrm{i}\mu n_0 \theta}{\exp(-\mathrm{i}\chi) + \omega'\mu}\right|^2 \tag{27}$$

and

$$(2q(2,0))^{1/2} - q(1,0)$$
$$= \left|\frac{\exp(-\mathrm{i}\chi) - \omega'\mu}{\exp(-\mathrm{i}\chi) + \omega'\mu}\right| (|2\tilde{u}^2 - 1| - 2|\tilde{u}|^2) \tag{28}$$
$$= \left(\frac{1 + \omega'^2\mu^2 - 2\omega'\mu \cos\chi}{1 + \omega'^2\mu^2 + 2\omega'\mu \cos\chi}\right)^{1/2} (|2\tilde{u}^2 - 1| - 2|\tilde{u}|^2).$$

Moreover, the following sum rule can be formulated:

$$\sum_{n'} q(n', 0) = \sum_m q(0, m) = 1/|T_0^0|^2$$
$$= \frac{1}{2}\left(\frac{1 + \omega'^2\mu^2 + 2\omega'\mu \cos\chi}{\omega'\mu \cos\chi}\right)^{1/2}$$
$$\times \exp\left(\frac{\omega' k(1 + \omega' k \cos\chi) d^2 - k n_0 d \omega' \mu 2\theta \sin\chi}{1 + \omega'^2\mu^2 + 2\omega'\mu \cos\chi}\right) \tag{29}$$

which may be useful for analysing the experimental data. It should be noted that the asymptotic behaviour of the coupling coefficients for large mode numbers $m, n' \gg 1$ can be investigated, too, and an analogue of the Franck-Condon principle may be formulated in this case. The procedure is quite similar to the one by Krivoshlykov and Sissakian (1979).

For the second coordinate (subscript 2), we have, according to (13), the same relations as before, only with $\theta = 0$ and $d_1 \equiv d_{x_1'} \to d_{x_2'}$.

6. Coaxial parabolic-index circular waveguides with gap between them

The modes of parabolic-index fibres with circular symmetry may conveniently be represented in a cylindrical coordinate system r, φ, z in terms of Gauss-Laguerre beams with plane wavefronts. During free space propagation, these modes transform into spherical Gauss-Laguerre beams near the front face of the second waveguide, which are described by a representation of generalised angular momentum states (see appendix 3). The coupling coefficients between these spherical Gauss-Laguerre beams and plane Gauss-Laguerre modes of the second waveguide give us the mode excitation coefficients. Unfortunately, the coupling coefficients between these Gauss-Laguerre beams can be expressed by known functions only in the case when the transverse offset and the tilt between the waveguides are absent. Therefore, we shall concentrate here on the problem of two coaxial circular parabolic-index waveguides with only a longitudinal gap between them and generalise the results obtained by Krivoshlykov and Sissakian (1979) for the case of waveguides without gap.

The mode coupling coefficients between these waveguides are given by the mode overlap integral $\langle v'l'|\underline{vl}\rangle$, where (cf (A3.5))

$$\langle r\varphi|v'l'\rangle=(-1)^{(v'+l')/2}\left\{\frac{\omega'k}{\pi}\left(\frac{v'-l'}{2}\right)!\left(\frac{v'+l'}{2}\right)!\right]^{-1}\Bigg\}^{1/2}(\omega'kr^2)^{l'/2}$$
$$\times\exp(-\omega'kr^2/2)L^{l'}_{(v'-l')/2}(\omega'kr^2)\exp(-il'\varphi) \tag{30}$$

are the Gauss–Laguerre modes of waveguide 2 and

$$\langle r\varphi|\underline{vl}\rangle=\left(\frac{k\cos\chi}{\pi\mu}\right)^{1/2}\left\{\left(\left(\frac{v-l}{2}\right)!\left(\frac{v+l}{2}\right)!\right]^{-1}\right\}^{1/2}[\exp(-i\chi)]^{v}(-1)^{(v+l)/2}$$
$$\times\left(\frac{k\cos\chi}{\mu}r^2\right)^{1/2}\exp\left(-\frac{k}{2\mu}\exp(-i\chi)r^2\right)L^{l}_{(v-l)/2}\left(\frac{k\cos\chi}{\mu}r^2\right)\exp(-il\varphi) \tag{31}$$

are the spherical Gauss–Laguerre modes at the front face of waveguide 2 irradiated by waveguide 1. The parameters μ and χ are connected with the gradient parameter ω and the gap z_0 between the waveguides according to (A1.16) and (14).

In order to calculate $\langle v'l|\underline{vl}\rangle$ (all matrix elements with $l\neq l'$ are vanishing), we may use the formula (Erdelyi 1954):

$$\int_0^\infty \exp(-x)x^a L^a_n(px)L^a_m(qx)\,dx$$
$$=(1-p)^n(1-q)^m\Gamma(a+1)\binom{n+a}{a}\binom{m+a}{a}$$
$$\times F\left(-n,-n;a+1;\frac{qp}{(1-p)(1-q)}\right) \tag{32}$$

and the following formula for Jacobi polynomials (Erdelyi 1953)

$$P_n^{(\alpha\beta)}(x)=\binom{n+\beta}{n}\left(\frac{x-1}{2}\right)^n F\left(-n,-n-\alpha;\beta+1;\frac{x+1}{x-1}\right). \tag{33}$$

Then it is easy to obtain the expression for the mode overlap integral in terms of Jacobi polynomials:

$$\langle v'l|\underline{vl}\rangle=(-1)^l(-1)^{(v+v')/2}[-\exp(-i\chi)]^v\left(\frac{2(\omega'\mu\cos\chi)^{1/2}}{\omega'\mu+\exp(-i\chi)}\right)^{l+1}$$
$$\times\left(\frac{[(v-l)/2]![(v'+l)/2]!}{[(v+l)/2]![(v'-l)/2]!}\right)^{1/2}\left(\frac{\omega'\mu-\exp(i\chi)}{\omega'\mu+\exp(-i\chi)}\right)^{(v-l)/2}$$
$$\times\left(\frac{\exp(-i\chi)-\omega'\mu}{\omega'\mu+\exp(-i\chi)}\right)^{(v'-l)/2}\left(\frac{2\omega'\mu\cos\chi+\omega'^2\mu^2+1}{2\omega'\mu\cos\chi-\omega'^2\mu^2-1}\right)^{(v-l)/2}$$
$$\times P^{((v'-l)/2,l)}_{(v-l)/2}\left(\frac{6\omega'\mu\cos\chi-\omega'^2\mu^2-1}{2\omega'\mu\cos\chi+\omega'^2\mu^2+1}\right). \tag{34}$$

The overlap integral between the fundamental mode of the first waveguide (spherical Gaussian beam) and modes of the second waveguide has the simple form

$$\langle v'0|\underline{00}\rangle=\left(\frac{2(\omega'\mu\cos\chi)^{1/2}}{\omega'\mu+\exp(-i\chi)}\right)\left(\frac{\omega'\mu-\exp(-i\chi)}{\omega'\mu+\exp(-i\chi)}\right)^{v'/2}. \tag{35}$$

In the case of excitation of a waveguide by a Gaussian beam with plane wavefront ($\chi=0$), or in the case of coupling between the waveguides without gap, the mode overlap integrals may be expressed in terms of Wigner's D functions (Krivoshlykov and Sissakian 1979). The recurrence relations for mode overlap integrals are similar to those obtained by Krivoshlykov and Sissakian (1979) and follow immediately from the recurrence relation for Jacobi polynomials (Erdelyi 1953).

7. Conclusions

We have investigated in a scalar theory the mode excitation of parabolic-index optical waveguides by spherical Gauss–Hermite and Gauss–Laguerre beams. As an example, the mode coupling between two waveguides possessing an infinite parabolic-index profile, with identical on-axis values $n(r=0)=n'(r=0)=n_0$ but different gradients $\omega \neq \omega'$ has been investigated. The imperfections of the connector consisted in a transversal shift d (offset), a tilt of the waveguide axes through the angle θ and an axial shift (gap) of the waveguides through the distance z_0. The algebraic procedure borrowed from quantum theory proved to be very convenient in deriving the recurrence relations and sum rules, which allow an easy calculation of mode coupling coefficients between arbitrary incoming and outgoing Gauss–Hermite modes. The parabolic profile represents a good approximation to other realistic index distributions. The results of this work are applicable to a lot of important practical problems, for example to a problem of fibre excitation by astigmatic elliptical Gauss–Hermite beams. Some of these problems will be taken up in forthcoming publications. We also hope that we can attack the taper problem in a similar manner.

Acknowledgments

This work was started when one of the authors (EGS) visited the Soviet Academy of Sciences in Moscow. He wants to express his sincerest thanks for the hospitality extended to him there by Professors Dr A M Prokhorov and Dr I N Sissakian. He also gratefully acknowledges very interesting discussions with Professor Dr V I Man'ko. This work was finished when the other author (SGK) visited the University of Karlsruhe. He wants to express his sincerest thanks to Professor Dr G K Grau for his kind invitation to visit the University of Karlsruhe, for hospitality and for extremely helpful discussions. He would also like to thank Dr W Freude for interesting discussions. Both authors thank both the Soviet Academy of Sciences and the Deutsche Forschungsgemeinschaft (DFG) for the financial support which made this exchange possible.

Appendix 1. Gaussian beams and coherent states

We consider an elliptical Gaussian beam (Yariv 1975)

$$\psi(x_1, x_2) \sim \prod_1^2 \exp\left(-ik\frac{x_j^2}{2q_j}\right) \tag{A1.1}$$

where q_j is the complex beam parameter with

$$\frac{1}{q_j} = \frac{1}{R_j} - \frac{2i}{kw_j^2}. \tag{A1.2}$$

Here, R_j is the radius of curvature in the x_jz plane and w_j is the beam width (for ψ) in this plane. These two parameters are z dependent.

(a) We consider such a z value (beam waist), where $R_j=\infty$, i.e. q_j is imaginary. Such a *Gaussian beam with plane wavefront* can be represented (Krivoshlykov and Sissakian 1979, 1980) by usual coherent states $|\alpha_1\alpha_2\rangle=|\boldsymbol{\alpha}\rangle=|\boldsymbol{\alpha}(\omega_j)\rangle$, which were constructed in quantum mechanics by Glauber (1963a, b) as eigenfunctions of the boson annihilation operators

$$\hat{a}_j=(k\omega_j/2)^{1/2}\left(\hat{x}_j+\frac{\mathrm{i}}{\omega_j}\hat{p}_j\right) \tag{A1.3}$$

that is,

$$\hat{a}_j|\alpha_1\alpha_2\rangle=\alpha_j|\alpha_1\alpha_2\rangle. \tag{A1.4}$$

$\hat{x}_j$ and $\hat{p}_j=-(\mathrm{i}/k)\partial_j$ are coordinate and momentum operators (with $k\to 1/\hbar$), whose eigenvalues x_j and p_j characterise the beam centre position and the beam slope (Marcuse 1972). ω_j is a positive parameter (cf (7), where we consider a circular-symmetric beam with $\omega_1=\omega_2=\omega$). The operators satisfy the following commutation relations:

$$[\hat{a}_i,\hat{a}_j^+]=\delta_{ij}\hat{I} \qquad [\hat{a}_i,\hat{a}_j]=0.$$

In coordinate representation the coherent state (A1.4) has the form

$$\psi_{\alpha_1\alpha_2}(x_1,x_2)=\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)=\langle x_1x_2|\alpha_1\alpha_2\rangle$$
$$=\prod_1^2\left(\frac{k\omega_j}{\pi}\right)^{1/4}\exp\left\{-\left[\left(\frac{k\omega_j}{2}\right)^{1/2}x_j-\alpha_j\right]^2+\frac{\alpha_j^2}{2}-\frac{|\alpha_j|^2}{2}\right\}. \tag{A1.5}$$

This is, indeed, a Gaussian beam with plane wavefront and width σ_j (standard deviation; $|\psi|^2$ considered), where

$$\sigma_j^2=\langle\boldsymbol{\alpha}|(\Delta\hat{x}_j)^2|\boldsymbol{\alpha}\rangle=1/(2k\omega_j)=w_j^2/4 \tag{A1.6}$$

with $\Delta\hat{x}_j=\hat{x}_j-\langle\hat{x}_j\rangle$. The complex eigenvalues α_j of the operator $\hat{a}_j$ (A1.3) are connected with the coordinates of the Gaussian beam centre in phase space by

$$\alpha_j=(k\omega_j/2)^{1/2}(x_j+(\mathrm{i}/\omega_j)p_j). \tag{A1.7}$$

The coherent states are not orthogonal

$$\langle\boldsymbol{\beta}|\boldsymbol{\alpha}\rangle=\exp(-\tfrac{1}{2}|\boldsymbol{\alpha}|^2-\tfrac{1}{2}|\boldsymbol{\beta}|^2+\boldsymbol{\alpha}\cdot\boldsymbol{\beta}^*) \tag{A1.8}$$

but form a complete (overcomplete) system of functions:

$$\pi^{-2}\int|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\,\mathrm{d}^2\boldsymbol{\alpha}=\hat{I}. \tag{A1.9}$$

Therefore, any field $|\boldsymbol{f}\rangle$ may be represented as a superposition of coherent states (A1.5), i.e. as a superposition of inhomogeneous plane waves with Gaussian amplitude:

$$|\boldsymbol{f}\rangle=\pi^{-2}\int|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|\boldsymbol{f}\rangle\,\mathrm{d}^2\boldsymbol{\alpha}. \tag{A1.10}$$

The value $|\langle\boldsymbol{\alpha}|\boldsymbol{f}\rangle|^2$ represents that part of the energy of the field $|\boldsymbol{f}\rangle$, which is transmitted by the given Gaussian beam $|\boldsymbol{\alpha}\rangle$.

It is interesting to note that coherent states (A1.5) minimise the usual uncertainty relation

$$\langle(\Delta\hat{x}_j)^2\rangle_\alpha\langle(\Delta\hat{p}_j)^2\rangle_\alpha = 1/4k^2 \tag{A1.11}$$

and show maximal localisation in phase space. Therefore, one can regard Gaussian beams (A1.5) as wavepackets, which are most close to geometrical rays (Krivoshlykov and Sissakian 1979, 1980). The centre of the beam gives the geometrical ray position in phase space and its width σ_j gives the region of ray field localisation.

(b) Now consider such a z value, where R_j is finite, i.e. q_j complex, representing a Gaussian beam with a spherical wavefront. Such a '*spherical Gaussian beam*' can be described (Krivoshlykov *et al* 1983a) by generalised (or correlated) coherent states $|\underline{\alpha_1\alpha_2}\rangle \equiv |\underline{\alpha}_1\underline{\alpha}_2\rangle = |\boldsymbol{\alpha}\rangle = |\boldsymbol{\alpha}(\mu_j, \chi_j)\rangle$, which were constructed in quantum mechanics (Dodonov *et al* 1980) as eigenfunctions of a generalised annihilation operator

$$\hat{\underline{a}}_j = \exp(\mathrm{i}\varphi_j)\left(\frac{k}{2\mu_j\cos\chi_j}\right)^{1/2}[\hat{x}_j + \mathrm{i}\mu_j\exp(\mathrm{i}\chi_j)\hat{p}_j] \tag{A1.12}$$

that is

$$\hat{\underline{a}}_j|\underline{\alpha_1\alpha_2}\rangle = \underline{\alpha}_j|\underline{\alpha_1\alpha_2}\rangle. \tag{A1.13}$$

Here, $|\chi_j| < \pi/2$, $\mu_j > 0$ and

$$[\hat{\underline{a}}_i, \hat{\underline{a}}_j^+] = \delta_{ij}\hat{I} \qquad [\hat{\underline{a}}_i, \hat{\underline{a}}_j] = 0.$$

Note while (A1.3) was a special linear combination of the operators $\hat{x}_j$, $\hat{p}_j$, (A1.12) represents the most general linear combination satisfying the given commutation relations.

In coordinate representation, the generalised coherent state (A1.13) has the form

$$\begin{aligned}\underline{\psi}_{\alpha_1\alpha_2}(x_1, x_2) &= \underline{\psi}_{\alpha_1}(x_1)\underline{\psi}_{\alpha_2}(x_2) = \langle x_1x_2|\underline{\alpha_1\alpha_2}\rangle \\ &= \prod_1^2\left(\frac{k\cos\chi_j}{\pi\mu_j}\right)^{1/4}\exp\left\{-\exp(-\mathrm{i}\chi_j)\left[\left(\frac{k}{2\mu_j}\right)^{1/2}x_j - (\cos\chi_j)^{1/2}\underline{\alpha}_j\exp(-\mathrm{i}\varphi_j)\right]^2\right. \\ &\quad \left. + \frac{\underline{\alpha}_j^2\exp(-2\mathrm{i}\phi_j)}{2} - \frac{|\underline{\alpha}_j|^2}{2}\right\}.\end{aligned} \tag{A1.14}$$

The complex parameter $\underline{\alpha}_j$ of the generalised coherent state (A1.13) is given by the position x_j and slope p_j of the Gaussian beam centre, analogously to the case of the usual coherent states (A1.5):

$$\boldsymbol{\alpha}_j = \exp(\mathrm{i}\varphi_j)\left(\frac{k}{2\mu_j\cos\chi_j}\right)^{1/2}[x_j + \mathrm{i}\mu_j\exp(\mathrm{i}\chi_j)p_j]. \tag{A1.15}$$

The parameters μ_j and χ_j of the generalised coherent states (A1.13) are connected with the beam width σ_j and the radius R_j (which may be positive or negative) of curvature of the Gaussian beam (A1.2) as follows:

$$\mu_j = \frac{2\sigma_j^2|R_j|}{(4\sigma_j^4 + R_j^2/k^2)^{1/2}} \qquad \sin\chi_j = -\frac{2\,\mathrm{sgn}\,R_j\sigma_j^2}{(4\sigma_j^4 + R_j^2/k^2)^{1/2}} \tag{A1.16a}$$

$$\sigma_j^2 = \frac{\mu_j}{2k\cos\chi_j} \qquad R_j = -\frac{\mu_j}{\sin\chi_j}. \tag{A1.16b}$$

φ_j is an arbitrary phase, which will be set equal to 0 in all our applications.

Generalised coherent states (A1.13) also possess the properties (A1.8)-(A1.10). Therefore, an arbitrary field $|f\rangle$ can be represented as a superposition of spherical Gaussian beams (A1.13).

The uncertainty relation for generalised coherent states (A1.13) has the form

$$\langle(\Delta\hat{x}_j)^2\rangle_{\alpha}\langle(\Delta\hat{p}_j)^2\rangle_{\alpha}=\frac{1}{4k^2\cos^2\chi_j}. \qquad \text{(A1.17)}$$

(c) All relations for the usual coherent states $|\alpha_1\alpha_2\rangle$ can be obtained from the corresponding ones for the generalised coherent states by specialising to the case $\chi_j=0$, $\varphi_j=0$ (and $\mu_j=1/\omega_j$). Both sets of states are unitarily equivalent:

$$\begin{aligned}&|\underline{\boldsymbol{\alpha}}(\mu_j,\chi_j)\rangle=\hat{S}|\boldsymbol{\alpha}(\omega_j)\rangle\\&\hat{\underline{a}}_j(\mu_j,\chi_j)=\hat{S}\hat{a}_j(\omega_j)\hat{S}^{+}\end{aligned} \qquad \text{(A1.18)}$$

with the unitary operator

$$\hat{S}=\exp\left\{-\tfrac{1}{4}\ln(\omega_j\mu_j\cos\chi_j)\left[\frac{-\mathrm{i}\omega_j\mu_j\sin\chi_j}{1-\omega_j\mu_j\cos\chi_j}(\hat{a}_j^{+}\hat{a}_j+\hat{a}_j\hat{a}_j^{+})\right.\right.$$
$$\left.\left.+\hat{a}_j^{2}\left(\frac{1-\omega_j\mu_j\exp(-\mathrm{i}\chi_j)}{1-\omega_j\mu_j\cos\chi_j}\right)+\hat{a}_j^{+2}\left(\frac{\omega_j\mu_j\exp(\mathrm{i}\chi_j)-1}{1-\omega_j\mu_j\cos\chi_j}\right)\right]\right\}. \qquad \text{(A1.19)}$$

The canonical transformation between the operators $\hat{\underline{a}}_j$ and $\hat{a}_j$ is given by

$$\hat{\underline{a}}_j(\mu_j,\chi_j)=u_j\hat{a}_j(\omega_j)+v_j\hat{a}_j^{+}(\omega_j) \qquad \text{(A1.20)}$$

where

$$\left.\begin{matrix}u_j\\v_j\end{matrix}\right\}=\frac{\exp(\mathrm{i}\varphi_j)}{2(\cos\chi_j)^{1/2}}[1/(\omega_j\mu_j)^{1/2}\pm\exp(\mathrm{i}\chi_j)(\omega_j\mu_j)^{1/2}] \qquad \text{(A1.21)}$$

and

$$|u_j|^2-|v_j|^2=1. \qquad \text{(A1.22)}$$

We write this linear homogeneous canonical transformation also in the form

$$\begin{pmatrix}\hat{a}_j\\\hat{a}_j^{+}\end{pmatrix}\to\begin{pmatrix}\hat{\underline{a}}_j\\\hat{\underline{a}}_j^{+}\end{pmatrix}=\begin{pmatrix}u_j&v_j\\v_j^{*}&u_j^{*}\end{pmatrix}\begin{pmatrix}\hat{a}_j\\\hat{a}_j^{+}\end{pmatrix}=T_1\begin{pmatrix}\hat{a}_j\\\hat{a}_j^{+}\end{pmatrix}. \qquad \text{(A1.23)}$$

Appendix 2. Gauss–Hermitian beams and Fock's representation of occupation numbers

(a) The *Gauss-Hermitian beams with plane wavefronts* correspond to Fock's representation of occupation numbers and may be constructed as eigenfunctions of the number operator $\hat{a}_j^{+}\hat{a}_j$:

$$\hat{a}_j^{+}\hat{a}_j|\boldsymbol{n}\rangle=n_j|\boldsymbol{n}\rangle \qquad n_j=0,1,2,\ldots \qquad \text{(A2.1)}$$

where $\hat{a}_j$ is given by (A1.3). They may also be constructed from the fundamental Gaussian beam $|00\rangle$ by

$$|n_1n_2\rangle=\frac{(\hat{a}_1^{+})^{n_1}(\hat{a}_2^{+})^{n_2}}{(n_1!\,n_2!)^{1/2}}|00\rangle. \qquad \text{(A2.2)}$$

It is known (Glauber 1963a, b) that the coherent states (A1.4) are generating functions for Fock's states (A2.1):

$$|\alpha_1\alpha_2\rangle = \exp\left(-\frac{|\alpha_1|^2+|\alpha_2|^2}{2}\right)\sum_{n_1,n_2=0}^{\infty}\frac{\alpha_1^{n_1}\alpha_2^{n_2}}{(n_1!\,n_2!)^{1/2}}|n_1n_2\rangle. \tag{A2.3}$$

Using (A1.5) in (A2.3), one can obtain an explicit expression for the Gauss-Hermitian modes (A2.1) in coordinate representation:

$$\begin{aligned}\psi_{n_1n_2}(x_1,x_2) &= \psi_{n_1}(x_1)\psi_{n_2}(x_2) = \langle x_1x_2|n_1n_2\rangle\\ &= \prod_1^2\left(\frac{k\omega_j}{\pi}\right)^{1/4}(2^{n_j}n_j!)^{-1/2}\exp\left(-\frac{k\omega_jx_j^2}{2}\right)H_{n_j}[(k\omega_j)^{1/2}x_j]\end{aligned} \tag{A2.4}$$

where H_n is an Hermite polynomial.

The modes $|n_1n_2\rangle$ form a complete and orthonormal set of functions:

$$\langle \boldsymbol{n}_i|\boldsymbol{n}_j\rangle = \delta_{ij} \qquad \sum_{n_1n_2}|n_1n_2\rangle\langle n_1n_2| = \hat{I}. \tag{A2.5}$$

Therefore, an arbitrary field $|f\rangle$ may be represented as a superposition of Gauss-Hermitian beams with plane wavefronts:

$$|f\rangle = \sum_{n_1n_2}|n_1n_2\rangle\langle n_1n_2|f\rangle. \tag{A2.6}$$

(b) We obtain the generalised Fock's representation of occupation numbers as eigenstates of the generalised number operator $\underline{\hat{a}}_j^+\underline{\hat{a}}_j$ (Krivoshlykov and Sissakian 1986):

$$\underline{\hat{a}}_j^+\underline{\hat{a}}_j|\underline{n_1n_2}\rangle = \underline{n}_j|\underline{n_1n_2}\rangle \qquad \underline{n}_1, \underline{n}_2 = 0, 1, 2, \ldots \tag{A2.7}$$

where $\underline{\hat{a}}_j$ is given by (A1.12). The equations (A2.2), (A2.3), (A2.5) and (A2.6) still hold, if we replace n_j, $\hat{a}_j$, α_j by the generalised (underlined) quantities. Therefore, we can obtain an expression for the generalised states (A2.7) using the generalised coherent states (A1.14) as generating functions for (A2.7) according to (A2.3):

$$\begin{aligned}\underline{\psi}_{n_1n_2}(x_1,x_2) &= \underline{\psi}_{n_1}(x_1)\underline{\psi}_{n_2}(x_2) = \langle x_1x_2|\underline{n_1n_2}\rangle\\ &= \prod_1^2\left(\frac{k\cos\chi_j}{\pi\mu_j}\right)^{1/4}\frac{\exp[-\mathrm{i}\underline{n}_j(\chi_j+\varphi_j)]}{(2^{\underline{n}_j}\underline{n}_j!)^{1/2}}\\ &\quad\times\exp\left(-\exp(-\mathrm{i}\chi_j)\frac{k}{2\mu_j}x_j^2\right)H_{\underline{n}_j}\left[\left(\frac{k\cos\chi_j}{\mu_j}\right)^{1/2}x_j\right].\end{aligned} \tag{A2.8}$$

It is easy to verify (Krivoshlykov and Sissakian 1986) that all '*spherical Gauss-Hermitian modes*' (A2.8) have the same radius R_j of wavefront curvature, which coincides with the radius of curvature of an axial Gaussian beam $|00\rangle$ (cf (A1.16)), and that the mode width σ increases with mode number $\underline{n}_j$ according to the formula

$$\sigma_j(\underline{n}_j) = \sigma_j(0)(2\underline{n}_j+1)^{1/2}. \tag{A2.9}$$

Appendix 3. Gauss–Laguerre beams and representation of generalised angular momentum states

The representation of generalised angular momentum states $|\underline{vl}\rangle$, which generalises the angular momentum representation $|vl\rangle$ in quantum mechanics, has been introduced by Krivoshlykov and Sissakian (1986), where it was shown that this representation may be used for the description of Gauss-Laguerre beams with axial symmetry and spherical wavefronts.

The beams with circular symmetry may conveniently be described in a cylindrical coordinate system r, φ, z, where $r \exp \mathrm{i}\varphi = x_1 + \mathrm{i}x_2$. Then the '*spherical Gauss-Laguerre modes*' $|\underline{vl}\rangle$ may be constructed as the simultaneous eigenstates of the generalised operators of occupation number

$$\hat{L}_0 = \hat{\underline{a}}_1^+ \hat{\underline{a}}_1 + \hat{\underline{a}}_2^+ \hat{\underline{a}}_2$$

and of angular momentum projection

$$\hat{L}_z = -\frac{\mathrm{i}}{k}\frac{\partial}{\partial\varphi} = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1 = \frac{\mathrm{i}}{k}(\hat{\underline{a}}_2^+ \hat{\underline{a}}_1 - \hat{\underline{a}}_1^+ \hat{\underline{a}}_2)$$

where the operators $\hat{\underline{a}}_j$ are given by the expression (A1.12) under the conditions $\mu_1 = \mu_2 = \mu$ and $\chi_1 = \chi_2 = \chi$, which correspond to circular beam symmetry. Thus, we have

$$\hat{L}_0|\underline{vl}\rangle = v|\underline{vl}\rangle \qquad \hat{L}_z|\underline{vl}\rangle = (l/k)|\underline{vl}\rangle \tag{A3.1}$$

where $v = m_1 + m_2$ is a non-negative integer, l is an integer, too, which can assume only the values $l = \pm v, \pm(v-2), \pm(v-4), \ldots, \pm 1$ or 0; $-v \leqslant l \leqslant v$ (v and l have equal parities).

Expressions for spherical modes $|\underline{vl}\rangle$ in cylindrical coordinates r, φ may be obtained using a procedure similar to that used by Malkin and Manko (1979).

Thus, the generalised coherent states $|\underline{\beta_1\beta_2}\rangle$, which are normalised eigenfunctions of the annihilation operators

$$\begin{aligned} &\hat{\underline{B}}_1 = (\hat{\underline{a}}_1 + \mathrm{i}\hat{\underline{a}}_2)/\sqrt{2} \qquad \hat{\underline{B}}_2 = (\hat{\underline{a}}_1 - \mathrm{i}\hat{\underline{a}}_2)/\sqrt{2} \\ &\hat{\underline{B}}_j|\underline{\beta_1\beta_2}\rangle = \underline{\beta}_j|\underline{\beta_1\beta_2}\rangle \end{aligned} \tag{A3.2}$$

should be used as the generating functions for spherical modes $|\underline{vl}\rangle$. The expression for the generalised coherent states $|\underline{\beta_1\beta_2}\rangle$ may be obtained from (A1.14), if we put $\underline{\alpha}_1 = (\underline{\beta}_1 + \underline{\beta}_2)/\sqrt{2}$ and $\underline{\alpha}_2 = (\underline{\beta}_1 - \underline{\beta}_2)/\mathrm{i}\sqrt{2}$. This expression in cylindrical coordinates has the form

$$\begin{aligned} \langle r, \varphi|\underline{\beta_1\beta_2}\rangle = &\left(\frac{k\cos\chi}{\pi\mu}\right)^{1/2} \exp\left\{-\frac{|\underline{\beta}_1|^2 + |\underline{\beta}_2|^2}{2} - \exp(-2\mathrm{i}\chi)\underline{\beta}_1\underline{\beta}_2\right\} \\ &\times \exp\left\{\exp(-\mathrm{i}\chi)\left[-\frac{k}{2\mu}r^2 + \left(\frac{k\cos\chi}{\mu}\right)^{1/2}\right.\right. \\ &\left.\left.\times r(\underline{\beta}_1 \exp(-\mathrm{i}\varphi) + \underline{\beta}_2 \exp(\mathrm{i}\varphi)\right]\right\}. \end{aligned} \tag{A3.3}$$

Then, using (A3.3) as generating function, according to the formula

$$|\underline{\beta_1\beta_2}\rangle = \exp(-|\underline{\beta}_1|^2/2)\exp(-|\underline{\beta}_2|^2/2)\sum_{v,l}\left(\frac{v+l}{2}!\frac{v-l}{2}!\right)^{-1/2} \underline{\beta}_1^{(v+l)/2}\underline{\beta}_2^{(v-l)/2}|\underline{vl}\rangle \tag{A3.4}$$

we obtain the expression for spherical Gauss-Laguerre modes $|\underline{vl}\rangle$ in cylindrical coordinates

$$\begin{aligned} \langle r\varphi|\underline{vl}\rangle = &\left(\frac{k\cos\chi}{\pi\mu}\right)^{1/2}\left\{\left(\frac{v-l}{2}!\right)\left[\left(\frac{v+l}{2}!\right)\right]^{-1}\right\}^{1/2} \exp(-\mathrm{i}l\varphi)[-\exp(-2\mathrm{i}\chi)]^{v/2}(-1)^{l/2} \\ &\times\left(\frac{k\cos\chi}{\mu}r^2\right)^{l/2} \exp\left(-\frac{k}{2\mu}\exp(-\mathrm{i}\chi)r^2\right) L^l_{(v-l)/2}\left(\frac{k\cos\chi}{\mu}r^2\right) \end{aligned} \tag{A3.5}$$

where $L_m^n(x)$ is the Laguerre polynomial. For beams with plane wavefront ($\chi = 0$), the spherical modes $|\underline{vl}\rangle$ transform to usual Gauss-Laguerre modes $|vl\rangle$ (see, for example, Snyder and Love 1983).

The spherical Gauss-Laguerre modes $|\underline{vl}\rangle$ as well as the usual modes $|vl\rangle$ form a complete and orthogonal set of functions

$$\sum_{v,l} |\underline{vl}\rangle\langle \underline{vl}| = \hat{I} \qquad \langle \underline{v_1 l_1}|\underline{v_2 l_2}\rangle = \delta_{v_1 v_2}\delta_{l_1 l_2}. \tag{A3.6}$$

Therefore, an arbitrary field $|f\rangle$ may be represented as a superposition of Gauss-Laguerre beams with spherical wavefronts

$$|f\rangle = \sum_{v,l} |\underline{vl}\rangle\langle \underline{vl}|f\rangle. \tag{A3.7}$$

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